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1983 J. Phys. A: Math. Gen. 16 L285

(http://iopscience.iop.org/0305-4470/16/9/001)

J. Phys. A: Math. Gen. **16** (1983) L285–L288. Printed in Great Britain

LETTER TO THE EDITOR

Lie symmetry, rational solution and bilinear operator structure for one- and two-dimensional nonlinear equations

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Received 23 March 1983

Abstract. We have systematised and extended the technique of Sym for obtaining the rational solutions of nonlinear equations in one and two space dimensions through the use of Lie symmetry and Hirota's bilinear operator. Our method is easily extensible for higher-dimensional integrable equations.

Recently Sym (1978) has demonstrated that it is possible to obtain a class of exact solutions of nonlinear evolution equations in one dimension through the use of Hirota's bilinear operator and assuming a solution of the form

$$f = t^b x^c \sum \bar{a}_n (\mu)^n, \qquad \mu = t^k x^l, \tag{1}$$

where f is connected to the nonlinear field u through $u = (\log f)_{xx}$ and a nonlinear equation $u_t = k(u)$ can be converted to the form

$$G(D_x, D_t) \cdot f \cdot f = 0 \tag{2}$$

where G, being a polynomial in D_x, D_t, is written as

$$G(D_x, D_t) = \sum a_{mn} D_t^m D_x^n \tag{3}$$

and D_x, D_t are Hirota's bilinear operator. Since each term in (2) will contain different powers in x and t the main problem is a suitable choice of the constants (k, l, b, c) so that it can be written again as a series in μ and, equating like powers of μ, we get a recursion relation for the coefficients $\bar{a}_n$. The mechanism for effecting this choice as suggested in Sym's paper is quite roundabout and is really difficult to implement. Here we propose that the most important choice of (k, l) comes automatically if we invoke Lie symmetry of the equation which yields the similarity variable x^δ/t^ν and we can set $\mu = x^\delta t^{-\nu}$ so that $k = \delta$, $l = -\nu$. We show below that this proposal works neatly in the cases of the higher-order KdV equation, Boussinesq equation and Sawada–Kotera equation.

(*i*) *Boussinesq equation* (*Ito 1982*)

The bilinear form is

$$(D_x^4 + 3D_t^2) f \cdot f = 0 \tag{4}$$

and the scaling law predicts

$$\mu = x^2 t^{-1} \tag{5}$$

so we set

$$f = \sum \bar{a}_n (x^2 t^{-1})^n.$$

(ii) Sawada–Kotera equation (Sato 1981)

The bilinear form is

$$(D_x^6 + 9D_x D_t) f \cdot f = 0 \tag{6}$$

and the Lie analysis yields

$$\mu = x^5 t^{-1} \tag{7}$$

so we set

$$f = \sum \bar{a}_n (x^5 t^{-1})^n.$$

(iii) Higher-order Ito equation (Sato 1981)

The bilinear version is

$$(D_t^2 + 2D_n^3 D_t) f \cdot f = 0. \tag{8}$$

Lie symmetry yields the scaling variable

$$\mu = x^{-3} t \tag{9}$$

so we put

$$f = \sum \bar{a}_n (x^{-3} t)^n.$$

Equation (2) reads as follows in each case which immediately yields the recursion relations ($c_i(ls, ls')$ are defined in Sym (1978)):

(i) $k = -1, l = 2$

$$(x^{-2}t^{-1})\Bigg[\sum_{s,s'} a_s a_{s'} (x^2 t^{-1})^{s+s'-1} c_4(ls, ls') c_0(ks, ks') + 3\sum_{s,s'} (x^2 t^{-1})^{s+s'+1} c_0(ls, ls') c_2(ks, ks')\Bigg] = 0$$

(ii) $k = -1, l = 5$

$$(x^{-1}t^{-1})\Bigg[\sum_{s,s'} a_s a_{s'} (x^5 t^{-1})^{s+s'-1} c_1(ks, ks') c_1(ls, ls') + \sum_{s,s'} (x^5 t^{-1})^{s+s'} c_0(ks, ks') c_6(ls, ls')\Bigg] = 0 \tag{10}$$

(iii) $k = 1, l = -3$

$$(x^{-3}t^{-1})\Bigg[\sum_{s,s'} a_s a_{s'} (x^{-3} t)^{s+s'} c_1(sk, sk') c_3(sl, s'l) + \sum_{s,s'} (x^{-3} t)^{s+s'-1} c_2(sk, s'k) c_0(sl, s'l) a_s a_{s'}\Bigg] = 0.$$

Since in our methodology we need not search for (k, l) in an arbitrary manner, we can extend the method easily to the many-dimensional case. Below we set out to elaborate the rule for the higher-dimensional cases.

Suppose the equation under consideration reads

$$u_t = k(u, u_x, u_y, u_z, \ldots). \tag{11}$$

Under the dependent variable transformation $u = (\log f)_{xx}$ let this be written in the form

$$H(D_x, D_t, D_y \ldots,)f \cdot f = 0 \tag{12}$$

where

$$H = \sum b_{lmn} \ldots D_x^l D_t^m D_y^n \ldots .$$

Now let us use appropriate similarity variables $\mu(x, t)$, $\sigma(x, y)\varepsilon(y, z) \ldots$ whose structures may be explicitly known from Lie–Bäcklund analysis. We then set

$$f = \sum A_{ijk} \ldots \mu^i(x, t)\sigma^j(x, y)\varepsilon^k(y, z) \tag{13}$$

and proceed à la Sym, so that the recurrence for the coefficients A_{ijk} can be obtained and solved. Below we give an example of the two space dimensions with the help of the Kadomtsev–Petviashvili equation (Date *et al* 1981). The equation is

$$\partial_x(u_t + 6uu_x + u_{xxx}) + \alpha u_{yy} = 0 \tag{14}$$

and can be written in Hirota's variables as

$$(D_x^4 + D_x D_t + D_y^2)f \cdot f = 0. \tag{15}$$

Now, it is known from the Lie-Bäcklund type (Tajiro *et al* 1982) analysis that this equation admits two scaling variables,

$$\mu(y, t) = yt^{-2/3}, \qquad \sigma(x, t) = xt^{-1/3}, \tag{16}$$

so we get

$$f = \sum f_{nm}\mu(y, t)^m \sigma(x, t)^n. \tag{17}$$

Then after the indicated operations of the D operators have been carried out we have

$$\begin{aligned}\sum f_{ij}f_{i'j'}[c_4(j, j')&(yt^{-2/3})^{i'+i}(xt^{-1/3})^{j'+j-3} \\ &+ c_1(j, j')c_1(i+j, i'+j')(yt^{-2/3})^{i'+i}(xt^{-1/3})^{j'+j} \\ &+ c_2(i, i')(yt^{-2/3})^{i'+i-2}(xt^{-1/3})^{j+j'+1}] = 0.\end{aligned} \tag{18}$$

This yields a recurrence for the coefficients a_{nm}. We obtain by equating coefficients of $(yt^{-2/3})^n(xt^{-1/3})^m$

$$f = (x^2 + \lambda^2 t^2 + y^2 + 2\lambda xt + yt + xy) \tag{19}$$

if we cut off the series (18) after the first two terms.

Two points may be mentioned at this point. Firstly the result (19) was obtained by Ablowitz and Satsuma (1978) by a limiting procedure from the general solution. Also, in the multidimensional case our solution (17) to some extent resembles the multiphase solution suggested recently by Flaschka and Newell (1981). From the analysis presented here it seems very encouraging to build up new types of solutions of further nonlinear equations because nowhere do we require the Lax pair for the inverse scattering transform.

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